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Numerical approximation of a degenerated non-conservative multifluid model: relaxation scheme‡

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SUMMARY

The present work is devoted to the numerical approximation of a system which arises when modelling a two-phase flow in a pipeline. Two particular difficulties are of special interest, the non-conservativity and the weakly hyperbolicity of this system. Some elementary waves are characterized and a relaxation system, unconditionally hyperbolic, is proposed. The stability criteria of the resulting relaxation method are achieved by a Chapmann–Enskog-like expansion. A numerical scheme based on the relaxation system is proposed and computations are performed on a shock tube. Validation is performed by comparison with the exact solution and also to the solution from a modified HLL scheme. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: multifluid model; finite volume method; relaxation scheme

1. INTRODUCTION

Let us consider the behaviour of a compressible fluid, essentially composed by two non-miscible components. This is for example the case of oil (index $k=1$) and water (index $k=2$) in pipelines. Each component (k) is characterized by: the void fraction $\alpha_k \in (0,1)$, the density $\rho_k > 0$, the velocity $u_k \in R$ and the pressure $p_k > 0$. According to the Baer–Nunziato model [1] and with the assumption of isentropic inviscid flow, the physical model is given by

$$\partial_t \mathbf{W} + \partial_x \mathbf{F}(\mathbf{W}) = \mathbf{B}(\mathbf{W})\partial_x \alpha_1 \tag{1}$$

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with

$$\mathbf{W}=\begin{pmatrix}\alpha_1\\ \alpha_1\rho_1\\ \alpha_1\rho_1u_1\\ \alpha_2\rho_2\\ \alpha_2\rho_2u_2\end{pmatrix},\quad \mathbf{F}(\mathbf{W})=\begin{pmatrix}0\\ \alpha_1\rho_1u_1\\ \alpha_1\rho_1u_1^2+\alpha_1p_1\\ \alpha_2\rho_2u_2\\ \alpha_2\rho_2u_2^2+\alpha_2p_2\end{pmatrix},\quad \mathbf{B}(\mathbf{W})=\begin{pmatrix}-u_I\\ 0\\ p_I\\ 0\\ -p_I\end{pmatrix} \tag{2}$$

where u_I and p_I are the interfacial velocity and pressure. We consider, as in Reference [1], that $u_I=u_1$ and $p_I=p_2$. The partial pressure p_k is a function of the partial density ρ_k ($p_k=p_k(\rho_k)$) and the sound speed is given by: $c_k=\sqrt{p'(\rho_k)/\rho_k}$.

This system is in a non-conservative form. It reduces to a conservative form in some specific cases (for example when one component is not present). The eigenvalues of (1) are real ($u_1, u_1\pm c_1$ and $u_2\pm c_2$) but the system is not diagonalizable when $\alpha_1=0$ or $u_1=u_2\pm c_2$. Although the system is non-conservative, we can characterize some elementary shocks by the following jump conditions:

$$[\alpha_1]=0\quad\text{and}\quad -\sigma[\mathbf{W}]+[\mathbf{F}(\mathbf{W})]=0 \tag{3}$$

where σ denotes the shock speed. The states $\mathbf{W}_0$ and $\mathbf{W}$ are connected by a contact discontinuity if and only if $\mathbf{W}$ belongs to the integral curve solution of: $d_\xi\mathbf{W}(\xi)=\mathbf{R}_0(\mathbf{W}(\xi))$ where $\mathbf{R}_0$ is the right eigenvector associated with the eigenvalue u_1.

From the numerical point of view, a precise evaluation of the interface localization is required. The HLL modified scheme [2] is too much diffusive. Indeed, after a long time calculation, the interface is spread on the entire domain. The schemes based on a complete spectral decomposition of the Riemann problem such as the VFRoe scheme [3], known to produce accurate approximation of the solutions, cannot be used when the system is not hyperbolic. In addition, the numerical approximation of solutions of a non-conservative system need a particular attention as soon as discontinuities appear [4].

2. THE RELAXATION METHOD

In this section we propose and analyse a relaxation system to design a numerical scheme. After the works of Chen *et al.* [5], Liu [6], Suliciu [7] and Coquel and Perthame [8], the relaxation method can be viewed as a well-established tool to approximate the solution of the compressible Euler equations of gas dynamics. Recently, Berthon *et al.* [9] have extended this method to a weakly hyperbolic system. The main feature of the relaxation solvers is to use a relaxation system for which the solution of the Riemann problem is easy to compute.

We consider a similar approach and we develop a relaxation model for the current non-conservative and weakly hyperbolic system. In this way, we introduce new variables Π_1 and Π_2 which are intended to coincide with the pressure p_1 and p_2 in the limit of the relaxation parameter λ. We propose the following relaxation system:

$$\partial_t\mathbf{V}+\tilde{A}(\mathbf{V})\partial_x\mathbf{V}=\frac{1}{\lambda}\mathbf{R}(\mathbf{V}) \tag{4}$$

with $\mathbf{V}^{\mathrm{t}} = (\alpha_1, \alpha_1\rho_1, \alpha_1\rho_1 u_1, \Pi_1, \alpha_2\rho_2, \alpha_2\rho_2 u_2, \Pi_2)$ and

$$\tilde{A}(\mathbf{V}) = \begin{pmatrix} u_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \Pi_1 - \Pi_2 & -u_1^2 & 2u_1 & \alpha_1 & 0 & 0 & 0 \\ 0 & -\frac{a_1^2 u_1}{\alpha_1 \rho_1^2} & \frac{a_1^2}{\alpha_1 \rho_1^2} & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -u_2^2 & 2u_2 & \alpha_2 \\ 0 & 0 & 0 & 0 & -\frac{a_2^2 u_2}{\alpha_2 \rho_2^2} & \frac{a_2^2}{\alpha_2 \rho_2^2} & u_2 \end{pmatrix} \quad \mathbf{R}(\mathbf{V}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ p_1 - \Pi_1 \\ 0 \\ 0 \\ p_2 - \Pi_2 \end{pmatrix} \tag{5}$$

where the positive relaxation parameters a_1 and a_2 are detailed later on.

Consider the set of admissible states: $\Omega = \{\mathbf{V} \in \mathbb{R}^7; \alpha_1 \in (0,1), \rho_1 > 0, \rho_2 > 0\}$. For all $\mathbf{V} \in \Omega$, the relaxation system (4) is unconditionally hyperbolic. Its eigenvalues v_i are u_1, $u_1 \pm (a_1/\rho_1)$, u_2, $u_2 \pm (a_2/\rho_2)$ and each field is linearly degenerated. The Riemann invariants associated with $v = u_k \pm (a_k/\rho_k)$ are: α_1, $u_k \pm (a_k/\rho_k)$, $\Pi_k \mp a_k u_k$, $\rho_{k'}, u_{k'}$, $\Pi_{k'}$ where k and k' denotes the two fluid components, with $v = u_1$ are: u_1, u_2, Π_2, $\alpha_2\rho_2$, $\alpha_1(\Pi_2 - \Pi_1)$ and with $v = u_2$ are: α_1, ρ_1, u_1, Π_1, u_2, Π_2.

Let us point out that the linear degeneracy of each characteristic fields of (5) is actually desirable since the property yields that the Riemann problem associated can be solved in a straightforward fashion for λ set to infinity.

According to Liu [6], Chen *et al.* [5] some compatibility conditions must be satisfied by the relaxation system (4). These conditions are actually needed to prevent numerical instabilities in the relaxation method when λ goes to zero. This conditions are usually referred to sub-characteristic like conditions after Whitham [10]. It is therefore expected that the parameters a_1 and a_2 must be fixed in order to fulfil such stability requirements. In the framework of our relaxation method it is convenient to follow the approach proposed by Chen *et al.* [5] or Whitham [10] based on the first-order asymptotic equilibrium system. In that way, we consider the Chapmann–Enskog expansion of small departures Π_1^λ, Π_2^λ from the equilibrium pressure p_1, p_2:

$$\Pi_1 = p_1 + \lambda \Pi_1^\lambda, \quad \Pi_2 = p_2 + \lambda \Pi_2^\lambda \tag{6}$$

After substituting (6) into (4) and neglecting higher order terms, we classically end up with the first-order asymptotic equilibrium system under the following generic form:

$$\partial_t \mathbf{W} + \partial_x \mathbf{F}(\mathbf{W}) - \mathbf{R}(\mathbf{W}) \partial_x \alpha_1 = \lambda \mathbf{C}(\mathbf{W}) \partial_x (\mathbf{D}(\mathbf{W}) \partial_x \mathbf{W})$$

$$\mathbf{C}(\mathbf{W}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_1 \rho_1} & 0 & \frac{-\alpha_2}{\alpha_1 \rho_1} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho_2} \end{pmatrix}, \quad \mathbf{D}(\mathbf{W}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_2 \delta & 0 & \alpha_1 \mu_1 & 0 & \alpha_2 \mu_2 \\ 0 & 0 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & \mu_2 \end{pmatrix} \tag{7}$$

with $\mu_1 = (1/\rho_1)(a_1^2 - \rho_1^2 p_1'(\rho_1))$, $\mu_2 = (1/\rho_2)(a_2^2 - \rho_2^2 p_2'(\rho_2))$ and $\delta = (\rho_2/\alpha_2)(u_1 - u_2)$.

Now, the stability conditions to be put on the pairs a_1, a_2 clearly come from the requirement that the first-order correction operator in (7) must be dissipative relatively to the zero-order approximation. Such conditions may be obtained by establishing the L^2-stability of the constant coefficient problem by linearizing (7) in the neighbourhood of the equilibrium pressure. In other words, for all admissible states, all eigenvalues of the matrix

$$-\mathrm{i}\xi A - \lambda\xi^2 CD \tag{8}$$

should have negative real parts [11]. After straightforward calculations, we find independent conditions for a_1 and a_2 given by

$$a_1 > \rho_1\sqrt{p_1'(\rho_1)}, \quad a_2 > \rho_2\sqrt{p_2'(\rho_2)} \tag{9}$$

3. NUMERICAL IMPLEMENTATION

The numerical scheme is based on the resolution of the Riemann problem associated to the relaxation system (4). This procedure is standard within the framework of the relaxation schemes [12, 13].

For the sake of completeness, we briefly recall the numerical relaxation procedure to approximate the weak solutions of (1).

We use a structured mesh in space and time defined by the cells $I_i = [x_{i-(1/2)}, x_{i+(1/2)}]$ and the time intervals $[t^n, t^{n+1}]$ with

$$t^n = n\Delta t \quad \text{and} \quad x_{i+(1/2)} = (i + \tfrac{1}{2})\Delta x \quad \text{with} \quad (n, i) \in \mathbb{N}\times\mathbb{Z} \tag{10}$$

where Δt denotes the time increment and Δx the spacial cell width.

We consider a piecewise constant approximation $\mathbf{W}^h$ of the equilibrium solution defined by

$$\mathbf{W}^h(x,t) = \mathbf{W}_i^n, \quad (x,t) \in I_i\times[t^n, t^{n+1}) \tag{11}$$

At the time $t = 0$, we set

$$\mathbf{W}_i^0 = \frac{1}{\Delta x}\int_{I_i} \mathbf{W}(x, t=0)\,\mathrm{d}x \tag{12}$$

The approximate solution at time t^n is then evolved to the next time level t^{n+1} by taking into account two steps:

(1) *Time evolution*: For $t^n < t < t^n + \Delta t$, we calculate an approximate solution $\mathbf{V}^{n+1,-}$ for system (5) with $\lambda \to +\infty$ for the initial data $\mathbf{V}^h(x, t^n)$ which is deduced from the equilibrium state $\mathbf{W}^n$ with $\Pi_1 = p_1(\rho_1)$ and $\Pi_2 = p_2(\rho_2)$. Under the CFL criterion

$$\frac{\Delta t}{\Delta x}\max|v_i(\mathbf{V}^h)| \leqslant \frac{1}{2} \tag{13}$$

the solution $\mathbf{V}^h(x, t^n + \Delta t)$ is composed of solutions of non-interacting Riemann problem. Let us note that each Riemann problem associated with the system is straightforward.

Next, the projection of $\mathbf{V}^h(x, t^n + \Delta t)$ on the piecewise constant functions is given by

$$\mathbf{V}_i^{n+1,-} = \frac{1}{\Delta x} \int_{x_{i-(1/2)}}^{x_{i+(1/2)}} \mathbf{V}^h(x, t^n + \Delta t)\, \mathrm{d}x \tag{14}$$

Let us emphasize that the relaxation model (4) is non-conservative. Thus, the Godunov scheme never can be written under the usual numerical conservative form (see Reference [13]).

(2) *Relaxation*: At time t^{n+1}, we define the updated approximate equilibrium solution $\mathbf{W}^{n+1}$ by computing:

$$(\alpha_k)_i^{n+1} = (\alpha_k)_i^{n+1,-},\ (\alpha_k \rho_k)_i^{n+1} = (\alpha_k \rho_k)_i^{n+1,-},\ (\alpha_k \rho_k u_k)_i^{n+1} = (\alpha_k \rho_k u_k)_i^{n+1,-} \tag{15}$$

where $i \in \mathbb{Z}, k = 1$ denotes the oil and $k = 2$ the water.

Setting $(\Pi_k)^{n+1} = p_k(\rho_k^{n+1})$, this step can also be viewed as the resolution of the ODE system:

$$\partial_t(\mathbf{V}) = \frac{1}{\lambda}\mathbf{R}(\mathbf{V}) \tag{16}$$

with the initial data $\mathbf{V}^{n+1,-}$ and λ set to zero (or equivalent to $t \to \infty$).

To conclude the present paper, we propose a numerical illustration of the above method. We use a uniform 1000 points mesh and the CFL number is fixed to 0.5. The above model is closed by the choice of the pressure laws: we consider $p_i = c_i^2 \rho_i$ where $c_1 = 200\,\mathrm{m/s}$ and $c_2 = 100\,\mathrm{m/s}$. We propose to approximate the solution of a shock tube. The initial data is made of two constant states separated by a discontinuity located at $x = 0.5$. The left and right states are given by

$$\alpha_1^{\mathrm{L}} = 0.9, \quad \rho_1^{\mathrm{L}} = 10, \quad u_1^{\mathrm{L}} = 0, \quad \rho_2^{\mathrm{L}} = 40, \quad u_2^{\mathrm{L}} = 0$$

$$\alpha_1^{\mathrm{R}} = 0.01, \quad \rho_1^{\mathrm{R}} = 17.1, \quad u_1^{\mathrm{R}} = 0, \quad \rho_2^{\mathrm{R}} = 59.8, \quad u_2^{\mathrm{R}} = 0$$

The exact solution is composed of a rarefaction wave and a shock wave separated by a contact discontinuity. The approximate solution is plotted Figure 1 and compared with the exact solution and the approximate solution obtained by the modified HLL scheme (see Reference [2]). The relaxation scheme gives better results and is less diffusive than the modified HLL scheme. The numerical approximation is in fairly good agreements with the exact solution. We obtain the expected velocity of the contact discontinuity while the HLL scheme introduce large errors in the position of the contact wave. In addition, the non-conservative numerical procedure leads to approximate shock waves according to the above definition (3). To conclude, the relaxation scheme detailed in the present paper is able to approximate weak solutions of the weakly hyperbolic system (1).

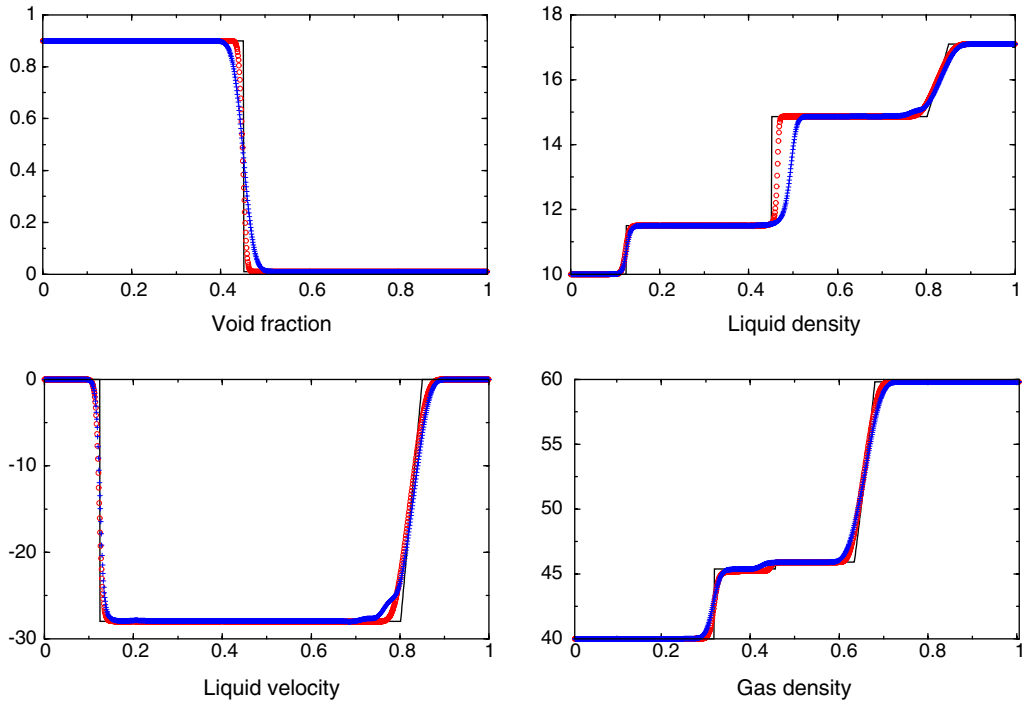

Figure 1. Shock tube problem, exact solution (fill line), relaxation scheme ($\circ$ symbols), HLL scheme ($+$ symbols), 1000 cells mesh, $t = 0.00175$, $c_1 = 200$, $c_2 = 100$, $\alpha_1^{\mathrm{L}} = 0.9$, $\rho_1^{\mathrm{L}} = 10$, $u_1^{\mathrm{L}} = 0$, $\rho_2^{\mathrm{L}} = 40$, $u_2^{\mathrm{L}} = 0$, $\alpha_1^{\mathrm{R}} = 0.01$, $\rho_1^{\mathrm{R}} = 17.1$, $u_1^{\mathrm{R}} = 0$, $\rho_2^{\mathrm{R}} = 59.8$, $u_2^{\mathrm{R}} = 0$.

REFERENCES

1. Baer MR, Nunziato JW. A two-phase mixture theory for the deflagration-to-detonation transition (DTR) in reactive granular materials. *International Journal of Multiphase Flows* 1986; **12**:861–889.
2. Massoni J, Saurel R, Nkonga B, Abgrall R. Proposition de méthodes et modèles euleriens pour les problèmes à interfaces entre fluides compressibles en présence de transfert de chaleur. *Journal of Heat and Mass Transfer* 2001; **45**(6):1287–1307.
3. Gallouet T, Masella J-M. Un schéma de Godunov approché. (French, English, French summary) [A rough Godunov scheme] *Comptes Rendus de l' Academie des Sciences Paris Série I-Mathematique* 1996; **323**(1):77–84.
4. Hou TY, LeFloch PG. Why nonconservative schemes converge to wrong solutions: error analysis. *Mathematics of Computation* 1994; **62**(206):497–530.
5. Chen GQ, Levermore CD, Liu T-P. Hyperbolic conservation laws with stiff relaxation terms and entropy. *Communications on Pure and Applied Mathematics* 1994; **47**(6):787–830.
6. Liu TP. Hyperbolic conservation laws with relaxation. *Communication in Mathematical Physics* 1987; **108**: 153–175.
7. Suliciu I. Energy estimates in rate-type thermo-viscoplasticity. *International Journal of Plasticity* 1998; **14**: 227–244.
8. Coquel F, Perthame B. Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics. *SIAM Journal on Numerical Analysis* 1998; **14**(6):2223–2249.
9. Berthon C, Breuss M, Titeux M-O. A relaxation scheme for the approximation of the pressureless Euler equations, submitted.
10. Whitham J. *Linear and Nonlinear Waves*. Wiley: New York, 1974.
11. Serre D. *Systems of Conservative Laws 1 and 2*. Cambridge University Press: Cambridge, 1999.
12. Jin S, Xin ZP. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Communications on Pure and Applied Mathematics* 1995; **48**:235–278.
13. LeVeque RJ. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press: Cambridge, 2002.

